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Non-Markovian dynamics of an interacting qubit pair coupled to two independent bosonic baths

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Abstract
The dynamics of two weakly interacting spins coupled to separate bosonic baths is studied. An analytical solution in the Born approximation for arbitrary spectral density functions of the bosonic environments is found. It is shown that in the non-Markovian cases concurrence 'lives' longer or reaches greater values.

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1. Introduction

The implementation of more and more efficient nanodevices exploitable in applicative contexts, like quantum computers, often requires a highly challenging miniaturization process aimed at packing a huge number of point-like basic elements, whose dynamics mimics indeed that of a qubit. Stimulated by such a requirement, over the last few years theoretical schemes have been investigated in the language of spin-$\frac{1}{2}$ models [1]. Apart from the simple dynamical behavior of each elementary constituent these Hamiltonian models do indeed capture the basic ingredients of several physical situations. In addition, spin models allow for the description of the effective interactions in a variety of different physical contexts ranging from high energy to nuclear physics [2, 3]. In condensed matter physics they capture several aspects of high-temperature superconductors, quantum Hall systems and heavy fermions [4–6]. We point out that Hamiltonians for interacting spins can be realized artificially in Josephson junctions arrays [7], with neutral atoms loaded in optical lattices [8–10] or with electrons in quantum dots [11].

In this context, a subject deserving a particular interest is entanglement dynamics. In view of possible applications it is important to understand the extent to which quantum coherences may be protected against the unavoidable degradation of the purity of the state, in particular in the presence of many-body interactions.

In this paper we focus our attention on a spin model recently introduced by Quiroga [12] and successively analyzed by other authors [13–16]. It consists of two interacting spins $\frac{1}{2}$, each coupled to a separate bosonic bath [12, 13]. Our aim is to study the entanglement dynamics of the two spins in the non-Markovian regime. Many authors have addressed the question of the dynamics of the entanglement between qubits in the non-Markovian environments. However, usually a system of non-interacting qubits in contact with separate bosonic baths is considered. Entanglement is either introduced in the initial preparation [17, 18] or created by the interaction of qubits with a common environment [19]. The focus of this paper is to study a system of directly interacting qubits. This is a typical situation in solid-state systems. For example, double quantum dots can be modeled as coupled qubit systems in contact with separate bosonic baths. For the demonstration of the dynamical properties of the system, in this paper we will consider Lorentz spectral density and Ohmic spectral density with a Lorentz–Drude cut-off. For a different model it has been shown that entanglement of qubits can occur in super-Ohmic environments even at non-vanishing temperature [20, 21].

The paper is structured as follows. In section 2 we describe in detail the model. In section 3 we present the analytical solution of the non-Markovian master equation for the reduced system constituted by the two interacting spins in the zero-temperature limit. In section 4 we analyze the entanglement dynamics of the two spins assuming for the environment a Lorentz spectral density and an Ohmic spectral density with the Lorentz–Drude cut-off function. Finally, conclusive remarks are given in section 5.

2. The model

Our analysis is focused on the dynamics of a composite system coupled to bosonic environments. Parts of the dynamical system are weakly interacting. The total Hamiltonian can be written as

$$H = H_S + \lambda^2 H_I + H_B + \lambda H_{SB}, \tag{1}$$

where $(H_S + \lambda^2 H_I)$ is the Hamiltonian describing the dynamics of the composite system, H_S is the Hamiltonian of the free components of the system, H_I is the Hamiltonian of interaction between the parts of the system. The operator H_B describes bosonic environments, and the Hamiltonian H_{SB} denotes the Hamiltonian of the interaction between the system and the environment. The parameter λ is a dimensionless expansion parameter. The non-Markovian dynamics of the reduced system will be described by a master equation containing terms no higher than the square of the expansion parameter λ.

The second-order time-convolutionless form of the master equation is given by [22]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S^I(t) = -\lambda^2 \int_0^t \mathrm{d}\tau \, \mathrm{tr_B}\big[H_{SB}(t), \big[H_{SB}(\tau), \rho_S^I(t) \otimes \rho_B\big]\big], \tag{2}$$

where $H_{SB}(t)$ denotes the Hamiltonian H_{SB} and $\rho_S^I(t)$denotes the density matrix of the reduced system in the interaction picture by the Hamiltonian $(H_S + \lambda^2 H_I + H_B)$. The density matrix $\rho_B = \mathrm{e}^{-\beta H_B}/\mathrm{tr}[\mathrm{e}^{-\beta H_B}]$ describes the state of the environment.

The present general approach is applied to a system consisting of a pair of weakly interacting spins, each coupled to a bosonic bath. The total Hamiltonian is given by equation (1). The Hamiltonian of the two free spins characterized by the same energy ϵ reads

$$H_S = \frac{\epsilon}{2}\sigma_1^z + \frac{\epsilon}{2}\sigma_2^z. \tag{3}$$

As usual σ_i^z and $\sigma_i^{\pm}$ are the Pauli operators describing the ith spin ($i = 1, 2$). The Hamiltonian of the weakly interacting spins is given by

$$\lambda^2 H_I = K\left(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\right), \tag{4}$$

where K is a constant defining the strength of the spin–spin interaction. The Hamiltonian of the bosonic baths characterized by the annihilation and creation operators b_{ni} and $b_{ni}^\dagger$ ($i = 1, 2$) reads

$$H_B = \sum_n \omega_{n,1} b_{n,1}^\dagger b_{n,1} + \sum_m \omega_{m,2} b_{m,2}^\dagger b_{m,2}. \tag{5}$$

The coupling of each spin to the separate bosonic baths is described by

$$H_{SB} = \sigma_1^+ \sum_n g_{n,1} b_{n,1} + \sigma_2^+ \sum_m g_{m,2} b_{m,2} + \text{h.c.}, \tag{6}$$

where $g_{n,1}$ and $g_{m,2}$ denote the coupling between the spin and its corresponding bosonic baths. In this paper units are chosen such that $k_B = \hbar = 1$. The Hamiltonian λH_{SB} in the interaction picture defined by the Hamiltonian $(H_S + \lambda^2 H_I + H_B)$ is given by

$$\lambda H_{SB}(t) = \sigma_1^+ \sum_n g_{n,1} b_{n,1}\, \mathrm{e}^{\mathrm{i}(\epsilon-\omega_{n,1})t} + \sigma_2^+ \sum_n g_{n,2} b_{n,2}\, \mathrm{e}^{\mathrm{i}(\epsilon-\omega_{n,2})t} + \text{h.c.} \tag{7}$$

In the above expression we neglect terms proportional to the cube of λ and higher. By direct calculation we show that

$$-\lambda^2 \int_0^t \mathrm{d}\tau\, \mathrm{tr_B}\left[H_{SB}(t), \left[H_{SB}(\tau), \rho_S^I(t) \otimes \rho_B\right]\right] = \sum_{j=1}^{2} \mathcal{L}^{(Dj)}(t)\rho_S^I(t), \tag{8}$$

where $\mathcal{L}^{(Dj)}(t)$ is the Liouville superoperator defined by

$$\begin{aligned}\mathcal{L}^{(Dj)}\rho_S^I(t) = {} & B^{(j)}(t)\left[\sigma_j^-\rho_S(t), \sigma_j^+\right] + \bar{B}^{(j)}(t)\left[\sigma_j^-, \rho_S(t)\sigma_j^+\right] \\ & + \bar{A}^{(j)}(t)\left[\sigma_j^+\rho_S(t), \sigma_j^-\right] + A^{(j)}(t)\left[\sigma_j^+, \rho_S(t)\sigma_j^-\right].\end{aligned} \tag{9}$$

The quantities $A^{(j)}(t)$ and $B^{(j)}(t)$ appearing in the previous expression are the so-called correlation functions whose explicit form is given by

$$\begin{aligned}A^{(j)}(t) &= \int_0^t \mathrm{d}\tau \sum_n |g_{n,j}|^2 \left\langle b_{n,j}^\dagger b_{n,j}\right\rangle_{Bj} \mathrm{e}^{\mathrm{i}(\epsilon-\omega_{n,j})(t-\tau)} \\ &= \mathrm{i} \sum_n |g_{n,j}|^2 \left\langle b_{n,j}^\dagger b_{n,j}\right\rangle_{Bj} \frac{1 - \mathrm{e}^{\mathrm{i}(\epsilon-\omega_{n,j})t}}{\epsilon - \omega_{n,j}},\end{aligned} \tag{10}$$

$$\begin{aligned}B^{(j)}(t) &= \int_0^t \mathrm{d}\tau \sum_n |g_{n,j}|^2 \left\langle b_{n,j} b_{n,j}^\dagger\right\rangle_{Bj} \mathrm{e}^{\mathrm{i}(\epsilon-\omega_{n,j})(t-\tau)} \\ &= \mathrm{i} \sum_n |g_{n,j}|^2 \left\langle b_{n,j} b_{n,j}^\dagger\right\rangle_{Bj} \frac{1 - \mathrm{e}^{\mathrm{i}(\epsilon-\omega_{n,j})t}}{\epsilon - \omega_{n,j}},\end{aligned} \tag{11}$$

where $\langle O\rangle_{Bj} \equiv \mathrm{tr}_{B_j}\{O\rho_{Bj}\}$, $\bar{A}^{(j)}(t)$ and $\bar{B}^{(j)}(t)$ being the complex conjugate of $A^{(j)}(t)$ and $B^{(j)}(t)$, respectively. To obtain expression (8) we used the fact that the bosonic environments assumed in this paper are uncorrelated with each other and $\left\langle b_{n,j} b_{n,j}^\dagger\right\rangle_{Bj}$, $\left\langle b_{n,j}^\dagger b_{n,j}\right\rangle_{Bj}$ are the only non-zero second-order correlations in the bath, all the other vanish.

Transforming back to the Schrödinger picture we obtain the following master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\mathrm{i}\left[\frac{\epsilon}{2}\sigma_1^z + \frac{\epsilon}{2}\sigma_2^z + K\left(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\right), \rho_S(t)\right] + \sum_{j=1}^{2} \mathcal{L}^{(Dj)}(t)\rho_S(t). \tag{12}$$

It is easy to see that the superoperator $\mathcal{L}_0$ defined as

$$\mathcal{L}_0\rho_S(t) = -\mathrm{i}\left[\frac{\epsilon}{2}\sigma_1^z + \frac{\epsilon}{2}\sigma_2^z, \rho_S(t)\right] \tag{13}$$

commutes with the superoperator $\mathcal{L}_{ME}(t)$ given by

$$\mathcal{L}_{ME}(t)\rho_S(t) = -\mathrm{i}\big[K\big(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\big), \rho_S(t)\big] + \sum_{j=1}^{2}\mathcal{L}^{(Dj)}(t)\rho_S(t), \tag{14}$$

and can be neglected as it is irrelevant for the dynamics of the expectation values defined by the density matrix $\rho_S(t)$. So, the final form of the master equation which is going to be studied in this paper reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\mathrm{i}\big[K\big(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\big), \rho_S(t)\big] + \sum_{j=1}^{2}\mathcal{L}^{(Dj)}(t)\rho_S(t). \tag{15}$$

3. Exact solution of the master equation

In order to solve the master equation (15), it is useful to separate the equations of motion for the diagonal elements of the density operator $\rho_S(t)$ from those relative to the off-diagonal elements. We have indeed proved that the diagonal and two non-diagonal elements of $\rho_S(t)$ have to satisfy the following system of the equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_S^{11}(t)\\ \rho_S^{22}(t)\\ \rho_S^{33}(t)\\ \rho_S^{44}(t)\\ \rho_S^{23}(t)\\ \rho_S^{32}(t)\end{pmatrix} = \Lambda_6(t)\begin{pmatrix}\rho_S^{11}(t)\\ \rho_S^{22}(t)\\ \rho_S^{33}(t)\\ \rho_S^{44}(t)\\ \rho_S^{23}(t)\\ \rho_S^{32}(t)\end{pmatrix}, \tag{16}$$

where

$$\Lambda_6(t) = \begin{pmatrix}-\beta_1-\beta_2 & \alpha_2 & \alpha_1 & 0 & 0 & 0\\ \beta_2 & -\alpha_2-\beta_1 & 0 & \alpha_1 & \mathrm{i}K & -\mathrm{i}K\\ \beta_1 & 0 & -\alpha_1-\beta_2 & \alpha_2 & -\mathrm{i}K & \mathrm{i}K\\ 0 & \beta_1 & \beta_2 & -\alpha_1-\alpha_2 & 0 & 0\\ 0 & \mathrm{i}K & -\mathrm{i}K & 0 & \xi & 0\\ 0 & -\mathrm{i}K & \mathrm{i}K & 0 & 0 & \bar{\xi}\end{pmatrix} \tag{17}$$

and

$$\begin{aligned}\alpha_j &= A^{(j)}(t) + \bar{A}^{(j)}(t), \qquad \beta_j = B^{(j)}(t) + \bar{B}^{(j)}(t),\\ \xi &= -A^{(1)}(t) - \bar{A}^{(2)}(t) - B^{(1)}(t) - \bar{B}^{(2)}(t).\end{aligned} \tag{18}$$

In what follows we will consider the case in which the two bosonic baths are both prepared in a thermal state with $T = 0$. This assumption in turn implies that the correlation functions reduce to

$$A^{(j)}(t) \equiv 0, \qquad B^{(j)}(t) \equiv B(t) = \mathrm{i}\sum_n |g_n|^2 \frac{1 - \mathrm{e}^{\mathrm{i}(\epsilon-\omega_n)t}}{\epsilon - \omega_n}. \tag{19}$$

Under these hypotheses it is possible to rewrite $\Lambda_6(t)$ in the following way: $\Lambda_6(t) = (B(t) + \bar{B}(t))L_1 + \mathrm{i}KL_2$, where L_1 and L_2 are 6×6 commuting matrices. Thus, the solution of the previous system of differential equations can be written as

$$\begin{pmatrix} \rho_S^{11}(t) \\ \rho_S^{22}(t) \\ \rho_S^{33}(t) \\ \rho_S^{44}(t) \\ \rho_S^{23}(t) \\ \rho_S^{32}(t) \end{pmatrix} = U_6(t) \begin{pmatrix} \rho_S^{11}(0) \\ \rho_S^{22}(0) \\ \rho_S^{33}(0) \\ \rho_S^{44}(0) \\ \rho_S^{23}(0) \\ \rho_S^{32}(0) \end{pmatrix}, \tag{20}$$

where

$$U_6(t) = \mathrm{T}\,\mathrm{e}^{\int_0^t \mathrm{d}\tau\, \Lambda_6(\tau)} = \mathrm{e}^{G(t)L_1}\,\mathrm{e}^{(\mathrm{i}Kt)L_2} \tag{21}$$

and the symbol T denotes the standard time ordering in the exponent. The function $G(t)$ appearing in the expression for the matrix $U^{(6)}(t)$ is defined as

$$G(t) = \Phi(t) + \bar{\Phi}(t), \tag{22}$$

with

$$\Phi(t) = \int_0^t \mathrm{d}\tau\, B(\tau) = \sum_n |g_n|^2 \frac{1 - \mathrm{e}^{\mathrm{i}(\epsilon-\omega_n)t} + \mathrm{i}(\epsilon - \omega_n)t}{(\epsilon - \omega_n)^2}. \tag{23}$$

The time dependence of the off-diagonal element $\rho_S^{14}(t)$ is trivial, namely $\rho_S^{14}(t) = \exp(-2\Phi(t))\rho_S^{14}(0)$. For the other off-diagonal elements we get the following system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \rho_S^{12}(t) \\ \rho_S^{13}(t) \\ \rho_S^{24}(t) \\ \rho_S^{34}(t) \end{pmatrix} = \Lambda_4(t) \begin{pmatrix} \rho_S^{12}(t) \\ \rho_S^{13}(t) \\ \rho_S^{24}(t) \\ \rho_S^{34}(t) \end{pmatrix}, \tag{24}$$

where

$$\Lambda_4(t) = \begin{pmatrix} -\beta - B(t) & \mathrm{i}K & 0 & 0 \\ \mathrm{i}K & -\beta - B(t) & 0 & 0 \\ 0 & \beta & -B(t) & -\mathrm{i}K \\ \beta & 0 & -\mathrm{i}K & -B(t) \end{pmatrix}. \tag{25}$$

One can check that the solution for the above equation has the following form:

$$\begin{pmatrix} \rho_S^{12}(t) \\ \rho_S^{13}(t) \end{pmatrix} = \mathrm{e}^{-G(t)-\Phi(t)}\,\mathrm{e}^{\mathrm{i}Kt\sigma_x} \begin{pmatrix} \rho_S^{12}(0) \\ \rho_S^{13}(0) \end{pmatrix} \tag{26}$$

and

$$\begin{pmatrix} \rho_S^{24}(t) \\ \rho_S^{34}(t) \end{pmatrix} = U_2(t) \begin{pmatrix} \rho_S^{24}(0) \\ \rho_S^{34}(0) \end{pmatrix} + U_2(t) \int_0^t \mathrm{d}\tau\, U_2^{-1}(\tau) \begin{pmatrix} \rho_S^{13}(\tau) \\ \rho_S^{12}(\tau) \end{pmatrix}, \tag{27}$$

where the operator $U_2(t)$ is defined by

$$U_2(t) = \mathrm{e}^{-\Phi(t)}\,\mathrm{e}^{-\mathrm{i}Kt\sigma_x}. \tag{28}$$

At this point we are in the position to explicitly write the density matrix of the two coupled spins at a generic time t starting from an arbitrary initial condition. For simplicity, we report on such a solution in the appendix. In what follows, instead, we focus on the cases in which the initial state of the pair of coupled spins is the Bell state $|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ or the factorized state $|\Psi_0\rangle = |10\rangle$. Exploiting the results presented in the appendix it is possible to demonstrate that the state of the reduced system at a generic time instant t can be written in the simple form

$$\rho_{S(\mathrm{Bell})}(t) = \mathrm{e}^{-G(t)}|\Psi_-\rangle\langle\Psi_-| + (1 - \mathrm{e}^{-G(t)})|00\rangle\langle 00| \tag{29}$$

and

$$\rho_{S(0)}(t) = \mathrm{e}^{-G(t)}|\Psi(t)\rangle\langle\Psi(t)| + (1 - \mathrm{e}^{-G(t)})|00\rangle\langle 00|, \tag{30}$$

where

$$|\Psi(t)\rangle = \cos(Kt)|10\rangle - \mathrm{i}\sin(Kt)|01\rangle. \tag{31}$$

Another point which we would like to mention here is the connection between the non-Markovian master equation (15) and the Markovian one. The Markovian limit of the master equation (15) can be constructed by taking the limit $t \to \infty$ in the set of correlation functions $A^{(j)}(t)$ and $B^{(j)}(t)$. The solution of the corresponding Markovian master equation for the system at hand can be constructed from non-Markovian ones by replacing functions $\Phi(t)$ and $G(t)$ with the corresponding Markovian ones

$$\Phi(t) \Rightarrow \Phi^M(t) = tB^M, \tag{32}$$

where

$$B^M = \lim_{t\to\infty} B(t). \tag{33}$$

In particular, for the function $G(t)$ we have

$$G(t) \Rightarrow G^M(t) = t(B^M + \bar{B}^M) = t2\pi J(\omega_0), \tag{34}$$

where $J(\omega_0)$ is the bath spectral density and $\omega_0 = \frac{\epsilon}{2}$.

4. Entanglement dynamics

As emphasized before, the solution we have found has been obtained without specifying the spectral properties of the bath. The density matrix $\rho_S(t)$ describing the pair of the coupled spins, however, depends on the bath spectral density through the function $G(t)$.

In this section, exploiting our results, we will analyze some dynamical properties of the central system for different spectral distributions of the environment. In particular, we will examine how the entanglement evolution is affected by the choice of the reservoir spectral density. Let us start by considering as a first case the Lorentzian distribution

$$J(\omega) = \frac{\gamma_0}{2\pi}\frac{\gamma^2}{\left(\omega - \frac{\epsilon}{2}\right)^2 + \gamma^2}, \tag{35}$$

where γ and γ_0 are the reservoir and the system decay rate respectively. This choice in turn implies that the correlation function $B(t)$, as given in the previous section, is

$$B(t) = \frac{\gamma_0}{2}(1 - \mathrm{e}^{-\gamma t}) \tag{36}$$

and consequently

$$G(t) = \gamma_0 t + \frac{\gamma_0}{\gamma}(\mathrm{e}^{-\gamma t} - 1). \tag{37}$$

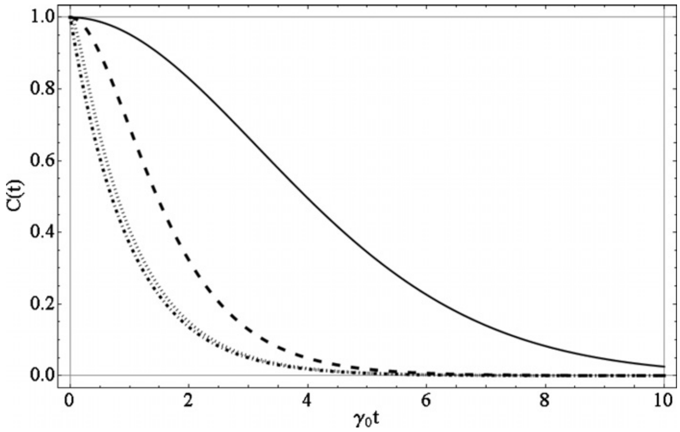

Figure 1. The concurrence $C(t)$ for a Lorentz bath distribution for different values of the ratio γ/γ_0 ($\gamma/\gamma_0 = 0.1$ (solid line), $\gamma/\gamma_0 = 1$ (dashed line), $\gamma/\gamma_0 = 10$ (dotted line), Markovian case (dot-and-dash line)). The initial state is the Bell state (38).

We have already demonstrated that starting from the Bell state

$$|\psi_-\rangle = \tfrac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \tag{38}$$

at a generic time t the density operator describing our system can be written as in equation (29). It is interesting to analyze how the interaction of the two coupled spins with the environments modifies the entanglement initially present in the system. To this end we consider the time behavior of the concurrence [23] of the two spins. Using equation (29) it is easy to demonstrate that in correspondence to any environmental spectral density, the concurrence is given by

$$C(t) = \mathrm{e}^{-G(t)}. \tag{39}$$

Thus, when $J(\omega)$ assumes the form (35), we have

$$C(t) = \exp\left(-(\gamma_0 t + \frac{\gamma_0}{\gamma}(\mathrm{e}^{-\gamma t} - 1))\right). \tag{40}$$

In figure 1 we plot $C(t)$ against $\gamma_0 t$ for different values of the ratio γ/γ_0 in the non-Markovian case. For comparison with the Markovian limit (34) we include also the Markovian case ($G^M(t) = \gamma_0 t$). As expected, in the presence of the two baths the concurrence function, starting from its maximum value, decreases as time elapses. However, in the non-Markovian regime, corresponding to $\gamma/\gamma_0 < 2$, the entanglement in the two spins persists for a longer time with respect to the Markovian case.

Suppose now that the two environments are characterized by an Ohmic spectral density with the Lorentz–Drude cut-off function [19, 24]:

$$J(\omega) = \frac{2\omega}{\pi}\frac{\omega_c^2}{\omega_c^2 + \omega^2}, \tag{41}$$

where ω is the frequency of the bath and ω_c is the cut-off frequency. Under this hypothesis, putting $\omega_0 = \epsilon/2$, the correlation function becomes

$$B(t) = -\mathrm{i}\frac{2\omega_c^2}{\omega_c - i\omega_0}(1 - \mathrm{e}^{-(\omega_c - i\omega_0)t}) \tag{42}$$

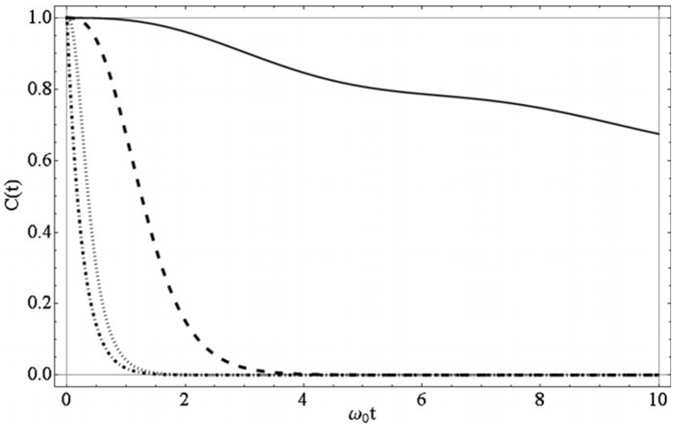

Figure 2. The concurrence $C(t)$ for a Lorentz–Drude bath distribution for different values of the ratio ω_c/ω_0 ($\omega_c/\omega_0 = 0.1$ (solid line), $\omega_c/\omega_0 = 1$ (dashed line), $\omega_c/\omega_0 = 10$ (dotted line), Markovian case (dot-and-dash line)). The initial state is the Bell state (38).

and thus

$$G(t) = 4\frac{\omega_c^2\omega_0}{\omega_c^2+\omega_0^2}t + 4\frac{\omega_c^2}{\left(\omega_c^2+\omega_0^2\right)^2}\left(\omega_c^2-\omega_0^2\right)\mathrm{e}^{-\omega_c t}\sin(\omega_0 t) \\ + 8\frac{\omega_c^3\omega_0}{\left(\omega_c^2+\omega_0^2\right)^2}\mathrm{e}^{-\omega_c t}\cos(\omega_0 t) - 8\frac{\omega_c^3\omega_0}{\left(\omega_c^2+\omega_0^2\right)^2}. \tag{43}$$

The corresponding Markovian function reads

$$G^M(t) = 2\pi J(\omega_0)t = 4\frac{\omega_c^2\omega_0}{\omega_c^2+\omega_0^2}t. \tag{44}$$

Using equation (39) it is possible to analyze the evolution of the degree of entanglement of the two spins starting from the Bell state (38). The results we have obtained are reported in figure 2 for different values of the ratio ω_c/ω_0. Comparing the four plots, we observe that when the spectrum of the reservoir does not completely overlap with the frequency of the system, that is $\omega_c \ll \omega_0$, the concurrence decreases to zero more slowly than in the opposite case, $\omega_c \gg \omega_0$. The results we have obtained, reported in figures 1 and 2, indicate that when the baths are characterized by Ohmic spectral densities with the Lorentz–Drude cut-off function, as given in equation (41), the entanglement initially present in the two spins can be preserved for a longer time with respect to the case of a Lorentz bath, at least for some values of the ratio ω_c/ω_0.

Following the analysis developed in this section it is also interesting to examine the behavior of the system starting from a factorized initial condition instead of an entangled one. In what follows, in particular, we suppose that at $t = 0$ the two spins are in the separable state $|1, 0\rangle$ and we study the time behavior of the concurrence. We find that in this case

$$C(t) = \mathrm{e}^{-G(t)}|\sin(2Kt)|. \tag{45}$$

The interaction between the two spins, as expressed by the effective Hamiltonian (4), enables the generation of entanglement starting from the factorized initial condition given before. On the other hand, in view of the fact that the two spins are coupled to two different baths, the quantum correlations that are established in the pair of spins will be destroyed. In the

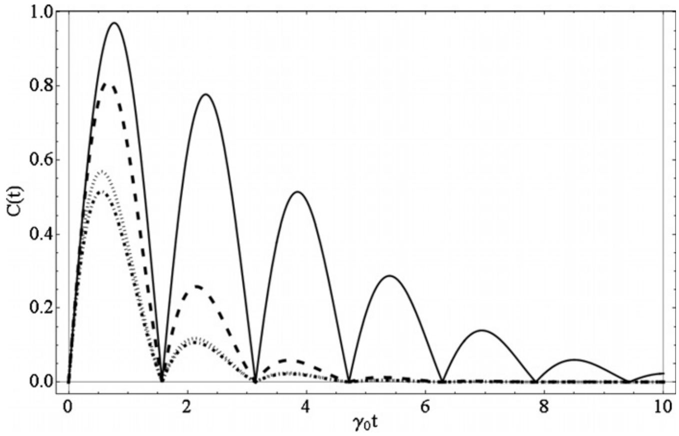

Figure 3. The concurrence $C(t)$ for a Lorentz bath distribution for different values of the ratio γ/γ_0 ($\gamma/\gamma_0 = 0.1$ (solid line), $\gamma/\gamma_0 = 1$ (dashed line), $\gamma/\gamma_0 = 10$ (dotted line), Markovian case (dot-and-dash line)). The initial state is $|1, 0\rangle$.

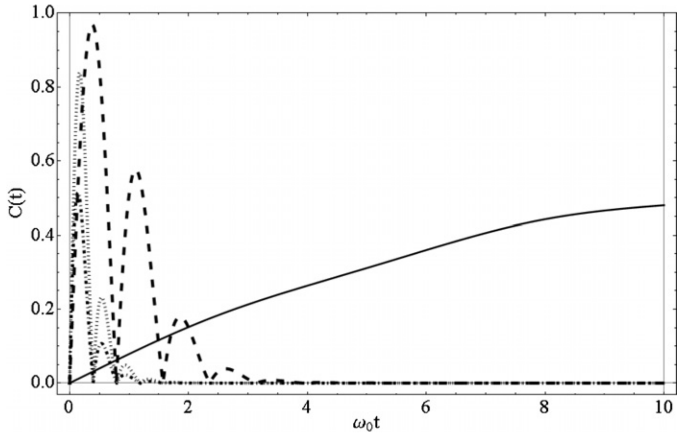

Figure 4. The concurrence $C(t)$ for a Lorentz–Drude bath distribution for different values of the ratio ω_c/ω_0 ($\omega_c/\omega_0 = 0.1$ (solid line), $\omega_c/\omega_0 = 1$ (dashed line), $\omega_c/\omega_0 = 10$ (dotted line), Markovian case (dot-and-dash line)). The initial state is $|1, 0\rangle$.

non-Markovian regime, however, we expect that the entanglement will be preserved for a longer time with respect to the Markovian one. This is confirmed by the time behavior of the concurrence function of the two spins for the Lorentzian spectral density of the baths (figure 3) and for the Ohmic spectral density of the baths (figure 4). Looking at these figures we also observe that the degree of entanglement that we can realize in the system starting from the state $|1, 0\rangle$ depends on the ratio γ/γ_0 or ω_c/ω_0. In particular, for the Lorentz spectral density, figure 3, the maximum value of the concurrence function is reached in the highly non-Markovian case, that is, $\gamma/\gamma_0 = 0.1$. For the Ohmic spectral density, figure 4, the highly non-Markovian case ($\omega_c/\omega_0 = 0.1$) corresponds to the presence of the quantum correlation in the system for the longest time.

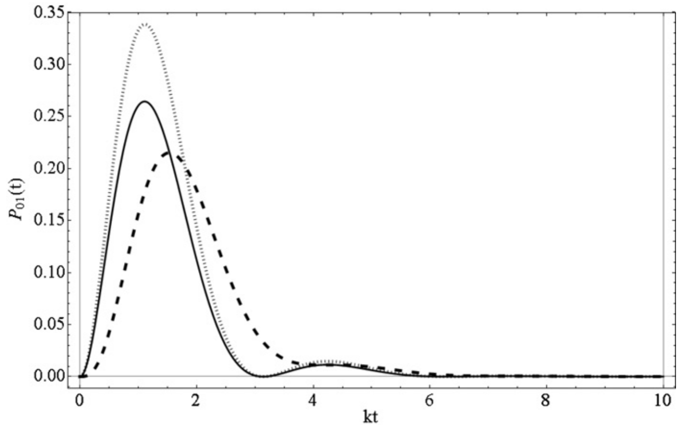

Figure 5. Dynamics of the probability of finding the system in the state $|0, 1\rangle$ (Markovian regime (solid line), post-Markovian regime (dashed line), non-Markovian regime (dotted line) for a Lorentz bath distribution with $\gamma/\gamma_0 = 4$). The initial state is the separable state $|1, 0\rangle$.

Before concluding we wish to compare our results with the ones obtained in the Markovian [13, 16] and post-Markovian [25, 26] regimes relatively to the same physical system. In order to do this, we concentrate our attention on the temporal behavior of the probability $P_{01}(t)$ of finding the qubit pair in the state $|0, 1\rangle$ supposing that at time $t = 0$ the system is prepared in the state $|1, 0\rangle$. In figure 5, where we show $P_{01}(t)$ in the three different regimes, time is scaled in units of the strength K of the spin–spin interaction. As shown, when we are in the non-Markovian regime, $P_{01}(t)$ reaches a maximum value that is greater than the one characterizing the Markovian and post-Markovian cases. Moreover, as expected in view of the presence of the two baths, in all the regimes the probability $P_{01}(t)$ decays toward zero after reaching its maximum value.

5. Conclusions

In this paper we have analyzed the non-Markovian dynamics of a pair of weakly interacting spins coupled to two separate bosonic baths. After deriving the second-order master equation, that is local in time, we have given an exact solution with the assumption that the two bosonic environments are both prepared in a thermal state with $T = 0$. It is important to emphasize that our solution is valid whatever the initial conditions of the system or the spectral properties of the two baths may be. From the solution of the non-Markovian master equation obtained we construct a solution of the corresponding master equation in the Markovian limit. Starting from the knowledge of the solution of the master equation we have studied the temporal behavior of the entanglement established in the pair of interacting spins for different spectral densities. The results show that in the non-Markovian case the concurrence, that is a measure of entanglement, of the system of two spins 'lives' longer or reaches greater values with respect to the Markovian regime. We wish to stress that the results presented in this paper are not directly connected to the so-called entanglement sudden death [27] because the concurrence does not vanish for a certain finite instant of time and has 'infinite' tails (39), (45). Our results motivate further studies on stronger coupling constants and non-zero temperatures.

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Appendix

The full solution for the density matrix of the pair of spins for arbitrary initial conditions reads:

$$\rho_S^{11}(t) = \mathrm{e}^{-2G(t)}\rho_S^{11}(0), \tag{A.1}$$

$$\begin{aligned}\rho_S^{22}(t) &= \mathrm{e}^{-G(t)}(1-\mathrm{e}^{-G(t)})\rho_S^{11}(0) + \mathrm{e}^{-G(t)}\cos^2(Kt)\rho_S^{22}(0)\\ &\quad + \mathrm{e}^{-G(t)}\sin^2(Kt)\rho_S^{33}(0) - \mathrm{e}^{-G(t)}\sin(2Kt)\,\mathrm{Im}\big(\rho_S^{23}(0)\big),\end{aligned} \tag{A.2}$$

$$\begin{aligned}\rho_S^{33}(t) &= \mathrm{e}^{-G(t)}(1-\mathrm{e}^{-G(t)})\rho_S^{11}(0) + \mathrm{e}^{-G(t)}\sin^2(Kt)\rho_S^{22}(0)\\ &\quad + \mathrm{e}^{-G(t)}\cos^2(Kt)\rho_S^{33}(0) + \mathrm{e}^{-G(t)}\sin(2Kt)\,\mathrm{Im}\big(\rho_S^{23}(0)\big),\end{aligned} \tag{A.3}$$

$$\rho_S^{44}(t) = 1 - \rho_S^{11}(t) - \rho_S^{22}(t) - \rho_S^{33}(t), \tag{A.4}$$

$$\begin{aligned}\rho_S^{23}(t) &= \mathrm{e}^{-G(t)}\cos^2(Kt)\rho_S^{23}(0) + \mathrm{e}^{-G(t)}\sin^2(Kt)\rho_S^{32}(0)\\ &\quad + \frac{\mathrm{i}}{2}\,\mathrm{e}^{-G(t)}\sin(2Kt)\big(\rho_S^{22}(0) - \rho_S^{33}(0)\big),\end{aligned} \tag{A.5}$$

$$\rho_S^{14}(t) = \mathrm{e}^{-2\Phi(t)}\rho_S^{14}(0), \tag{A.6}$$

$$\rho_S^{12}(t) = \mathrm{e}^{-G(t)-\Phi(t)}\cos(Kt)\rho_S^{12}(0) + \mathrm{i}\,\mathrm{e}^{-G(t)-\Phi(t)}\sin(Kt)\rho_S^{13}(0), \tag{A.7}$$

$$\rho_S^{13}(t) = \mathrm{e}^{-G(t)-\Phi(t)}\cos(Kt)\rho_S^{13}(0) + \mathrm{i}\,\mathrm{e}^{-G(t)-\Phi(t)}\sin(Kt)\rho_S^{12}(0), \tag{A.8}$$

$$\begin{aligned}\rho_S^{24}(t) &= \mathrm{e}^{-\Phi(t)}\cos(Kt)\rho_S^{24}(0) - \mathrm{i}\,\mathrm{e}^{-\Phi(t)}\sin(Kt)\rho_S^{34}(0)\\ &\quad + \int_0^t \mathrm{d}\tau\beta(\tau)\,\mathrm{e}^{-G(\tau)}\big(\cos K(t-\tau)\rho_S^{13}(\tau) - \mathrm{i}\sin K(t-\tau)\rho_S^{12}(\tau)\big),\end{aligned} \tag{A.9}$$

$$\begin{aligned}\rho_S^{34}(t) &= \mathrm{e}^{-\Phi(t)}\cos(Kt)\rho_S^{34}(0) - \mathrm{i}\,\mathrm{e}^{-\Phi(t)}\sin(Kt)\rho_S^{24}(0)\\ &\quad + \int_0^t \mathrm{d}\tau\beta(\tau)\,\mathrm{e}^{-G(\tau)}\big(\cos K(t-\tau)\rho_S^{12}(\tau) - \mathrm{i}\sin K(t-\tau)\rho_S^{13}(\tau)\big).\end{aligned} \tag{A.10}$$

We are going to show that the solution of the non-Markovian master equation is positive. For simplicity we assume an arbitrary X-like initial state of the two-qubit system;

$$\begin{aligned}\rho_S(0) &= p_0|00\rangle\langle 00| + p_1|01\rangle\langle 01| + p_2|10\rangle\langle 10| + (1-p_0-p_1-p_2)|11\rangle\langle 11|\\ &\quad + C_{12}|01\rangle\langle 10| + \bar{C}_{12}|10\rangle\langle 01| + C_{03}|00\rangle\langle 11| + \bar{C}_{03}|11\rangle\langle 00|.\end{aligned} \tag{A.11}$$

The function $G(t)$ can be rewritten in the following way:

$$G(t) = \Phi(t) + \bar{\Phi}(t) = 4\sum_n |g_n|^2 \frac{\sin^2\frac{(\epsilon-\omega_n)t}{2}}{(\epsilon-\omega_n)^2} \geqslant 0. \tag{A.12}$$

After straightforward transformations we get

$$\begin{aligned}\rho_S^{44}(t) &= \big(1 - \rho_S^{11}(0) - \rho_S^{22}(0) - \rho_S^{33}(0)\big) + (1-\mathrm{e}^{-G(t)})^2\rho_S^{11}(0)\\ &\quad + (1-\mathrm{e}^{-G(t)})\big(\rho_S^{22}(0) + \rho_S^{33}(0)\big);\end{aligned} \tag{A.13}$$

taking into account the above expression for $\rho_S^{44}(t)$ and the fact that $G(t) \geqslant 0$ it is obvious that $\rho_S^{11}(t)$ and $\rho_S^{44}(t)$ are nonnegative. To prove the positivity of the solution we need to show that $\rho_S^{22}(t)$ and $\rho_S^{33}(t)$ are nonnegative too. To this end we show that

$$\cos^2(Kt)\rho_S^{22}(0) + \sin^2(Kt)\rho_S^{33}(0) - \sin(2Kt)\,\mathrm{Im}\big(\rho_S^{23}(0)\big) \geqslant 0. \qquad \text{(A.14)}$$

Using the positivity condition for the initial density matrix $\rho_S(0)$ which implies that $p_1p_2 \geqslant |C_{12}|^2$ or $\rho_S^{22}(0)\rho_S^{33}(0) \geqslant \left|\rho_S^{23}(0)\right|^2$ we can strengthen the above inequality by replacing $\sin(2Kt)\,\mathrm{Im}\big(\rho_S^{23}(0)\big)$ with $\pm\sin(2Kt)\sqrt{\rho_S^{22}(0)\rho_S^{33}(0)}$ and get

$$\begin{aligned}\cos^2(Kt)\rho_S^{22}(0) &+ \sin^2(Kt)\rho_S^{33}(0) \pm \sin(2Kt)\sqrt{\rho_S^{22}(0)\rho_S^{33}(0)}\\ &= \left(\cos(Kt)\sqrt{\rho_S^{22}(0)} \pm \sin(Kt)\sqrt{\rho_S^{33}(0)}\right)^2 \geqslant 0. \qquad \text{(A.15)}\end{aligned}$$

Thus, from the above inequality it follows that $\rho_S^{22}(t) \geqslant 0$. The same statement for $\rho_S^{33}(t)$ is established analogously. This proves that the density matrix is positive.

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